J. Fluid Mech. (1978), *vol.* 87, *part* 2, *pp.* 233–241
Printed in Great Britain

On the stability of flow in an elliptic pipe which is nearly circular

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(Received 18 November 1977)

The linear stability of Poiseuille flow in an elliptic pipe which is nearly circular is examined by regarding the flow as a perturbation of Poiseuille flow in a circular pipe. We show that the temporal damping rates of non-axisymmetric infinitesimal disturbances which are concentrated near the wall of the pipe are decreased by the ellipticity. In particular we estimate that if the length of the minor axis of the cross-section of the pipe is less than about $96\frac{1}{2}\%$ of that of the major axis then the flow will be unstable and a critical Reynolds number will exist. Also we calculate estimates of the ellipticities which will produce critical Reynolds numbers ranging from 1000 upwards.

1. Introduction

Hocking (1977) has considered the stability of steady flow in a straight pipe of elliptic cross-section when the ellipticity e is close to 1, so that the length of the major axis of the cross-section is much larger than that of the minor axis. For such a pipe the flow in the central region near the minor axis is almost plane Poiseuille flow between the slightly curved boundaries at the ends of the minor axis. Moreover the stability characteristics of the flow are dominated by this central region so they may be obtained, as Hocking showed, by a perturbation away from plane Poiseuille flow. With a Reynolds number based on the length of the minor axis, he found that the critical Reynolds number increased as e decreased below 1 by an amount proportional to $(1-e)^{\frac{1}{2}}$. This increase in the critical Reynolds number is only to be expected in view of the fact that as e decreases towards zero the pipe becomes more and more circular and it is generally accepted that circular pipe flow is stable to infinitesimal disturbances.

The situation, then, is that elliptic pipe flow is linearly unstable if e is close to 1 whereas it is stable if e is equal to 0. It seems likely therefore that there exists a critical value of e above which the flow will have a critical Reynolds number and below which the flow will be stable and there will be no such critical Reynolds number. The aim of this paper is to estimate this critical value of e and to verify that when e is small the effect of the ellipticity is to decrease the damping rate of an infinitesimal disturbance. So we approach the problem from the opposite end of the spectrum to that considered by Hocking, i.e. we take e to be small and determine the linear stability characteristics of the flow by a perturbation from circular pipe flow. In what follows this procedure seems to be justified since the estimate which we obtain for the critical value of e is quite small.

Linear stability theory for circular pipe flow is somewhat complicated in the sense that both axisymmetric and non-axisymmetric disturbances must be considered and by the fact that when the Reynolds number is large a disturbance may be concentrated either near the centre of the pipe, a 'centre' mode, or near the wall of the pipe, a 'wall' mode, or may be distributed across the pipe if it has a sufficiently small axial wave-number, a 'distributed' mode. Since we shall perturb away from circular pipe flow we must be careful to consider the effect of the ellipticity on all these types of disturbance. Indeed we shall find that the key disturbance for the elliptic problem is a wall mode, whereas the least-damped disturbance for the circular problem is a distributed mode as pointed out by Gill (1973), who called a distributed mode an l-mode.

2. Linear stability of flow in a pipe with small ellipticity

The steady flow whose stability we wish to examine is that of a viscous incompressible fluid flowing along a straight pipe of elliptic cross-section under the action of a constant pressure gradient. The length of the semi-minor axis of the elliptic cross-section is a and the length of the semi-major axis is b, so that $a \leqslant b$ and the ellipticity e is defined by

$$a^2 = b^2(1-e^2). \tag{1}$$

We suppose that the centre-line speed of Poiseuille flow along the pipe is U_0, so that the constant pressure gradient needed to maintain the flow is

$$-2\rho\nu U_0\left(\frac{1}{a^2}+\frac{1}{b^2}\right),$$

where ν is the kinematic viscosity of the fluid and ρ is its density.

We choose a, U_0 and a/U_0 as the characteristic scales of length, speed and time, respectively, with respect to which we make our quantities non-dimensional. The reference pressure is $\rho\nu U_0/a$. We use non-dimensional Cartesian co-ordinates (x,y,z) with the x axis in the direction of the basic flow down the pipe and with the y and z axes in the directions of the major and minor axes respectively, so that the boundary of the pipe is

$$(1-e^2)\,y^2+z^2 = 1 \tag{2}$$

and the steady flow down the pipe is given by

$$U = 1-(1-e^2)\,y^2-z^2. \tag{3}$$

We shall also use polar co-ordinates (r,θ) in the cross-sectional plane defined by

$$y = r\cos\theta, \quad z = r\sin\theta, \tag{4}$$

so that (x,r,θ) are cylindrical polars and hence the boundary of the pipe (2) may be written as

$$r^2 = (1-e^2\cos^2\theta)^{-1} \tag{5}$$

and also (3) becomes

$$U = 1-r^2-e^2r^2\cos^2\theta. \tag{6}$$

We shall consider only linear stability theory and suppose that a disturbance will grow or decay temporally without spatial modulation. Hence it suffices to express the fluid velocity $U_0(u_x, u_r, u_\theta)$ and the pressure $(\rho\nu U_0/a)\mathscr{P}$ in the form

$$\left.\begin{aligned} u_x &= U+\epsilon Eu(r,\theta)+O(\epsilon^2),\\ u_r &= \epsilon Ev(r,\theta)+O(\epsilon^2),\\ u_\theta &= \epsilon Ew(r,\theta)+O(\epsilon^2),\\ \mathscr{P} &= P+\epsilon Ep(r,\theta)+O(\epsilon^2). \end{aligned}\right\} \quad (7)$$

In (7), ϵ is a measure of the amplitude of the disturbance compared with that of the basic flow and

$$E \equiv \exp\{i\alpha(x-ct)\}, \quad (8)$$

so that the disturbance has wavenumber α in the x direction, wave speed c_r and temporal growth rate αc_i, where

$$c = c_r + ic_i. \quad (9)$$

Also it should be understood that complex conjugates are to be added to the right-hand sides of (7) to cancel the imaginary terms.

If (7) and (8) are substituted into the Navier–Stokes equations and the continuity equation then the terms of order ϵ yield

$$\left.\begin{aligned} \left(\mathscr{L}+\frac{1}{r^2}\right)u-i\alpha p &= i\alpha Re^2r^2\cos^2\theta u-2Rr(1-e^2\cos^2\theta)\,v-Re^2r\sin 2\theta w,\\ \mathscr{L}v-\frac{\partial p}{\partial r} &= i\alpha Re^2r^2\cos^2\theta v+\frac{2}{r^2}\frac{\partial w}{\partial\theta},\\ \mathscr{L}w-\frac{1}{r}\frac{\partial p}{\partial\theta} &= i\alpha Re^2r^2\cos^2\theta w-\frac{2}{r^2}\frac{\partial v}{\partial\theta} \end{aligned}\right\} \quad (10)$$

and

$$\frac{\partial v}{\partial r}+\frac{v}{r}+i\alpha u+\frac{1}{r}\frac{\partial w}{\partial\theta} = 0, \quad (11)$$

where $R = U_0 a/\nu$ is the Reynolds number and the operator $\mathscr{L}$ is defined by

$$\mathscr{L} \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\alpha^2-i\alpha R(1-r^2-c). \quad (12)$$

The boundary conditions are

$$\left.\begin{aligned} &u = v = w = 0 \quad \text{when} \quad r = (1-e^2\cos^2\theta)^{-\frac{1}{2}},\\ &\frac{\partial u}{\partial\theta} = \frac{\partial v}{\partial\theta}-w = \frac{\partial w}{\partial\theta}+v = \frac{\partial p}{\partial\theta} = 0 \quad \text{when} \quad r = 0. \end{aligned}\right\} \quad (13)$$

We now expand the variables u, v, w and p and the eigenvalue c in powers of e^2 as follows:

$$u = u_0+e^2u_1+e^4u_2+\ldots, \quad (14a)$$

$$v = v_0+e^2v_1+e^4v_2+\ldots, \quad (14b)$$

$$w = w_0+e^2w_1+e^4w_2+\ldots, \quad (14c)$$

$$p = p_0+e^2p_1+e^4p_2+\ldots, \quad (14d)$$

$$c = c_0+e^2c_1+e^4c_2+\ldots\ . \quad (14e)$$

If we substitute these expansions into (10)–(13), clearly the leading-order terms u_0, v_0, w_0, p_0 and c_0 are just the solution of the linear stability equations for circular pipe flow. Hence

$$\{u_0, v_0, w_0, p_0\} = \{\bar{u}_0(r), \bar{v}_0(r), \bar{w}_0(r), \bar{p}_0(r)\} \exp(in\theta), \tag{15}$$

where n is an integer and $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$, $\bar{p}_0$ and c_0 are the solutions of the linear ordinary differential Orr–Sommerfeld system

$$\begin{pmatrix} L+1/r^2 & 2Rr & 0 & -i\alpha \\ 0 & L & -2in/r^2 & -D \\ 0 & 2in/r^2 & L & -in/r \\ i\alpha & D+1/r & in/r & 0 \end{pmatrix} \begin{pmatrix} \bar{u}_0 \\ \bar{v}_0 \\ \bar{w}_0 \\ \bar{p}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{16}$$

where $D \equiv d/dr$ and the operator L is defined by

$$L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{1+n^2}{r^2} + \alpha^2\right) - i\alpha R(1 - r^2 - c_0). \tag{17}$$

The boundary conditions for (16) and (17) are

$$\left.\begin{aligned} \bar{u}_0 = \bar{v}_0 = \bar{w}_0 = 0 \quad \text{when} \quad r = 1, \\ in\bar{u}_0 = in\bar{v}_0 - \bar{w}_0 = in\bar{w}_0 + \bar{v}_0 = in\bar{p}_0 = 0 \quad \text{when} \quad r = 0; \end{aligned}\right\} \tag{18}$$

also $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$ and $\bar{p}_0$ must be finite as $r \to 0$. Equation (16) subject to the boundary conditions (18) can be solved numerically to determine $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$, $\bar{p}_0$ and the eigenvalue c_0 for an arbitrary choice of the integer n; see, for example, Lessen, Sadler & Lin (1968) and also Salwen & Grosch (1972). Note that the boundary condition at $r = 0$ takes rather special forms when $n = 0$ or 1, but that otherwise it reduces to

$$\bar{u}_0 = \bar{v}_0 = \bar{w}_0 = \bar{p}_0 = 0.$$

In general the least-damped disturbances are those with $n = 1$ (but see Gill 1973), in which case the boundary condition at $r = 0$ becomes $\bar{u}_0 = \bar{v}_0 + i\bar{w}_0 = \bar{p}_0 = 0$, with $\bar{v}_0$ and $\bar{w}_0$ finite.

From (10), (11) and (14) at order ϵ^2 we have

$$\left.\begin{aligned} \left(\mathscr{L}_0 + \frac{1}{r^2}\right) u_1 + 2Rrv_1 - i\alpha p_1 &= \{i\alpha(r^2\cos^2\theta - c_1)\bar{u}_0 + 2r\cos^2\theta\bar{v}_0 - r\sin 2\theta\bar{w}_0\} R\, e^{in\theta}, \\ \mathscr{L}_0 v_1 - \frac{2}{r^2}\frac{\partial w_1}{\partial\theta} - \frac{\partial p_1}{\partial r} &= i\alpha R(r^2\cos^2\theta - c_1)\,\bar{v}_0 e^{in\theta}, \\ \mathscr{L}_0 w_1 + \frac{2}{r^2}\frac{\partial v_1}{\partial\theta} - \frac{1}{r}\frac{\partial p_1}{\partial\theta} &= i\alpha R(r^2\cos^2\theta - c_1)\,\bar{w}_0 e^{in\theta} \end{aligned}\right\} \tag{19}$$

and

$$\frac{\partial v_1}{\partial r} + \frac{v_1}{r} + i\alpha u_1 + \frac{1}{r}\frac{\partial w_1}{\partial\theta} = 0, \tag{20}$$

where the operator $\mathscr{L}_0$ is the same as $\mathscr{L}$ [see (12)], but with c replaced by c_0. Also, from (13) and (14) the boundary conditions for u_1, v_1, w_1 and p_1 are

$$\left.\begin{aligned} (u_1, v_1, w_1) = -\tfrac{1}{2}\cos^2\theta(\bar{u}_0', \bar{v}_0', \bar{w}_0')\, e^{in\theta} \quad \text{when} \quad r = 1, \\ \frac{\partial u_1}{\partial\theta} = \frac{\partial v_1}{\partial\theta} - w_1 = \frac{\partial w_1}{\partial\theta} + v_1 = \frac{\partial p_1}{\partial\theta} = 0 \quad \text{when} \quad r = 0, \end{aligned}\right\} \tag{21}$$

where a prime denotes differentiation with respect to r.

The θ dependence of the forcing terms in (19) and (21) may be written as linear combinations of $\exp\{in\theta\}$, $\exp\{i(n+2)\theta\}$ and $\exp\{i(n-2)\theta\}$ so we can seek a solution for u_1, v_1, w_1 and p_1 which is a linear combination of these three quantities. When we seek that part of the solution which is proportional to $\exp\{in\theta\}$, the partial differential operator for the homogeneous part of (19) becomes the same as the ordinary differential operator in (16) and so has $\{u_0, v_0, w_0, p_0\}$ as eigensolutions. Hence we can use adjoint theory to determine c_1 and thus find the leading-order effect of the ellipticity on the rate of decay of the disturbance.

From (19) and (20) it follows that the vector equation for that part of the solution for u_1, v_1, w_1 and p_1 which is proportional to $\exp\{in\theta\}$ is

$$\begin{pmatrix} L+1/r^2 & 2Rr & 0 & -i\alpha \\ 0 & L & -2in/r^2 & -D \\ 0 & 2in/r^2 & L & -in/r \\ i\alpha & D+1/r & in/r & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ p_1 \end{pmatrix} = i\alpha R(\tfrac{1}{2}r^2 - c_1) \begin{pmatrix} \bar{u}_0 \\ \bar{v}_0 \\ \bar{w}_0 \\ \bar{p}_0 \end{pmatrix} + \begin{pmatrix} Rr\bar{v}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{22}$$

and this equation will have a solution only if the right-hand side is orthogonal to the solution space of the left-hand operator. In order to determine this condition we must find the solution of the associated adjoint problem

$$\begin{pmatrix} L+1/r^2 & 0 & 0 & i\alpha \\ 2Rr & L & 2in/r^2 & -D \\ 0 & -2in/r^2 & L & in/r \\ -i\alpha & D+1/r & -in/r & 0 \end{pmatrix} \begin{pmatrix} u_a \\ v_a \\ w_a \\ p_a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{23}$$

with boundary conditions

$$\left.\begin{aligned} u_a = v_a = w_a = 0 \quad \text{when} \quad r = 1, \\ inu_a = inv_a + w_a = inw_a - v_a = inp_a = 0 \quad \text{when} \quad r = 0. \end{aligned}\right\} \tag{24}$$

Also, if $n = 0$ or $n = 1$ then u_a, v_a, w_a and p_a must be finite as $r \to 0$.

We solve (23) and (24) numerically and check that the new value which we obtain for c_0 is close to the corresponding value found from the Orr–Sommerfeld formulation (16)–(18). Having found u_a, v_a, w_a and p_a we now left-multiply (22) by

$$\{ru_a, rv_a, rw_a, rp_a\}^{\mathrm{T}}$$

and integrate from $r = 0$ to $r = 1$ to obtain

$$c_1 = \frac{\dfrac{1}{2}\displaystyle\int_0^1 r^3(\bar{u}_0 u_a + \bar{v}_0 v_a + \bar{w}_0 w_a)\,dr - \frac{i}{\alpha}\int_0^1 r^2 \bar{v}_0 u_a\,dr + \frac{i}{4\alpha R}[\bar{u}_0' u_a' + \bar{w}_0' w_a']_{r=1}}{\displaystyle\int_0^1 r(\bar{u}_0 u_a + \bar{v}_0 v_a + \bar{w}_0 w_a)\,dr}, \tag{25}$$

so that the temporal growth rate of the disturbance is given by

$$\begin{aligned} \sigma &= \mathscr{R}\{-i\alpha(c_0 + e^2 c_1)\} + O(e^4) \\ &= \alpha c_{0i} + e^2 \alpha c_{1i} + O(e^4). \end{aligned} \tag{26}$$

Thus to leading order the effect of the ellipticity on the growth rate is determined by the value of c_1 as calculated from (25). We expect that c_{1i} will be positive, so that the effect of the ellipticity can lead to instability.

Once c_1 has been found there is no difficulty in principle in successively calculating higher-order terms in (14a–e), such as the coefficient c_2 of the term of order e^4 in (14e). In practice, however, this is a formidable task because the number of differential equations which must be solved numerically increases by an order of magnitude every time an extra term in each of (14a–e) is required.

3. Numerical results

Numerical solutions of the ordinary differential equations (16) and (23) were obtained by Runge–Kutta integration using a shooting method in a very similar manner to that described by Garg & Rouleau (1972), although we also used orthonormalization so that we could do calculations for much larger values of αR.

To integrate the Orr–Sommerfeld system (16), which is of seventh order, we first reduced it to a sixth-order system by differentiating the fourth row of (16) and subtracting it from the second row so that the second row is replaced by

$$-\bar{p}_0' = \left\{\frac{n^2}{r^2}+\alpha^2+i\alpha R(1-r^2-c_0)\right\}\bar{v}_0+i\left(\alpha\bar{u}_0'+\frac{n\bar{w}_0'}{r}+\frac{n\bar{w}_0}{r^2}\right). \tag{27}$$

Initially, near $r = 0$, it is necessary to use a power-series expansion before the Runge–Kutta integration is begun. When $n \neq 0$ the leading terms in the expansion are of the form

$$\left.\begin{aligned} \bar{u}_0 &= r^n(a_1+a_2r^2+\ldots),\\ \bar{v}_0 &= r^{n-1}(b_1+b_2r^2+\ldots),\\ \bar{w}_0 &= r^{n-1}(d_1+d_2r^2+\ldots),\\ \bar{p}_0 &= r^n(f_1+f_2r^2+\ldots),\end{aligned}\right\} \tag{28}$$

where $d_1 = ib_1$ and a_2, b_2, d_2, f_2 and the coefficients of the higher-order terms can easily be found as linear functions of a_1, b_1 and f_1. The solutions are therefore characterized by the three quantities a_1, b_1 and f_1 so three integrations are begun from some small value of r with $\{a_1, b_1, f_1\} = \{1, 0, 0\}$, $\{0, 1, 0\}$ and $\{0, 0, 1\}$. The three solutions generated by these starting conditions are then orthonormalized when necessary. The procedure is similar for $n = 0$ although then (16) reduces to a fourth-order system and only two independent solutions need be generated and orthonormalized.

To integrate the adjoint system (23) we also first reduce it from seventh to sixth order by differentiating the continuity equation with respect to r and subtracting it from the second row of (23) so that this row is replaced by

$$p_a' = -\left\{\frac{n^2}{r^2}+\alpha^2+i\alpha R(1-r^2-c_0)\right\}v_a+2Rru_a+i\left\{\alpha u_a'+\frac{nw_a'}{r}+\frac{nw_a}{r^2}\right\}. \tag{29}$$

A similar expansion to (28) is used for r small and then the procedure is just as described above for the Orr–Sommerfeld system. [If $n \neq 0$ then $d_1 = -ib_1$ in the expansion corresponding to (28).]

Before we discuss the numerical results in detail let us recall that our principal aim is to obtain an estimate for the critical value of e, i.e. the value above which there will be a critical Reynolds number but below which the flow will be stable. A necessary preliminary to achieving this aim is to determine, given a large fixed value of R, for which type of disturbance the effect of the ellipticity will be most likely to lead to instability.

To answer this question, for each value of R shown in table 1 we calculated c_0 and c_1 for a very wide range of values of α, for $n = 0, 1, 2$ and several larger values and for both the first few centre modes, wall modes and distributed modes. For each pair of values (R, n) we then found the smallest possible value of $-c_{0i}/c_{1i}$ as α was varied, each kind of mode (centre, wall and distributed) being considered. It is these values of $-c_{0i}/c_{1i}$ which are shown in table 1.

The reason why we are particularly interested in the smallest possible value of $-c_{0i}/c_{1i}$ is as follows: given R and n, if we ignore the terms of order e^4 in (26) then a disturbance will be unstable if $c_{0i} + e^2 c_{1i} > 0$, i.e. if $e > e_{\min}$, where, as a first approximation, we define

$$e^2_{\min} \equiv \text{smallest possible value of } -c_{0i}/c_{1i} \text{ for } R, n \text{ fixed.} \tag{30}$$

Since $c_{0i} < 0$ we need $c_{1i} > 0$ for $e_{\min}$ to be real and we find that this is always the case.

The first fact which we were able to establish numerically was that for all values of n the centre modes are only very slightly affected by the ellipticity of the boundary. This is presumably because the centre modes are concentrated near the centre of the pipe so one would not expect them to be affected very much by the slight distortion of a distant boundary. (Nevertheless the distortion of the boundary does affect the curvature of the velocity profile of the basic steady flow near the centre-line so this argument is not entirely convincing.) Since the wall modes are concentrated near the boundary they will be affected much more by the ellipticity than the centre modes. All the values of $-c_{0i}/c_{1i}$ found for the centre modes are so very large that we shall not discuss these modes any further, even though they are less damped than the wall modes for *circular* pipe flow.

Next we found that for the axisymmetric case $n = 0$ the values of $-c_{0i}/c_{1i}$ for both the wall modes and the distributed modes are large, so that the ellipticity has only a small effect. For this case the smallest possible values of $-c_{0i}/c_{1i}$ are produced by the distributed mode which is associated with the least-damped wall mode (see figure 5 of Gill 1965) and, although they are large, we include them in table 1 for illustrative purposes.

The competition to be the mode which is most likely to lead to instability lies therefore between the wall modes and the distributed modes for $n = 1$ and $n = 2$. We did many calculations for larger values of n but found that none of the modes with $n > 2$ were as important as $n = 2$.

Table 1 contains the principal results which we obtained for the cases $n = 1$ and $n = 2$. The columns headed $-c_{0i}/c_{1i}$ are the smallest possible values so these columns could also have been headed $e^2_{\min}$ as defined by (30); the columns headed α are the corresponding wavenumbers. The first four rows for the case $n = 1$ and the first row for the case $n = 2$ are for the distributed mode which is associated with the least-damped wall mode, while the remaining entries for these cases are for the least-damped wall mode.

For each value of n table 1 indicates that $e_{\min}$ is a monotone decreasing function of R with a finite limit as $R \to \infty$. This means that, given R and n, the flow will be unstable if $e > e_{\min}$. Alternatively it means that, given n and $e = e_{\min}$, R will be the critical

	$n = 1$		$n = 2$		$n = 0$	
R	α	$-c_{0i}/c_{1i}$	α	$-c_{0i}/c_{1i}$	α	$-c_{0i}/c_{1i}$
1000	0·408	0·11428	0·559	0·15167	0·298	1·754
2000	0·223	0·10296	0·607	0·12950	0·149	1·748
5000	0·117	0·09743	0·462	0·11123	0·060	1·747
10000	0·061	0·09584	0·364	0·10314	0·030	1·746
20000	0·252	0·08724	0·274	0·09819		
50000	0·179	0·07686	0·167	0·09491		
100000	0·123	0·07279	0·100	0·09397		
150000	0·093	0·07153	0·071	0·09375		
200000	0·074	0·07099	0·054	0·09366	$300R^{-1}$	1·746
250000	0·061	0·07071	0·044	0·09362		
300000	0·052	0·07054	0·037	0·09360		
∞		0·0701		0·0935		

TABLE 1. For given R and n we show the smallest possible value of $-c_{0i}/c_{1i} = e^2_{\text{min}}$ as defined by (30) and the corresponding value of α; to a first approximation the flow is unstable if $e > e_{\text{min}}$. Whether an entry is for a wall mode or for a distributed mode is indicated in the text.

Reynolds number and α will be the critical wavenumber at the 'nose' of the neutral-stability curve. Note that when R is large α is also a monotone decreasing function of R and the critical wavelengths are much larger than the mean radius of the pipe. The limiting values of $-c_{0i}/c_{1i}$ as $R \to \infty$ were found from the convergence of algebraic and exponential Shanks transforms.

In general we found that the contributions to c_{1i} from the two integrals in the numerator of (25) nearly cancelled each other, so that the last term in the numerator dominates the value of c_{1i}. However the first integral dominates the value of c_{1r}, which is positive, so that one effect of the ellipticity is to increase the phase speed. For example, if $n = 1$, $R = 100\,000$ and $\alpha = 0{\cdot}123$, then $c_0 = 0{\cdot}2336 - 0{\cdot}0300i$, $c_1 = 0{\cdot}4033 + 0{\cdot}4116i$ and the three contributions to c_1 from (25) are, in order, $0{\cdot}4407 + 0{\cdot}0851i$, $-0{\cdot}0294 - 0{\cdot}0757i$ and $-0{\cdot}0081 + 0{\cdot}4022i$.

It is clear from table 1 that we are concerned with small values of e^2 and this gives us some justification for neglecting the terms of order e^4 in (26); typically we find that $|c_1|$ is less than twice $|c_0|$, so that if e^2 is of order 0·1 then (26) probably converges quite rapidly. Nevertheless it is not really valid to ignore these terms and we should remember that (30) gives us only a first approximation to the relation which must hold between (R, n) and the ellipticity for a neutral disturbance to exist. What we can say with certainty is that since c_{1i} is always positive the effect of the ellipticity is to make the flow less stable. For large values of αR each coefficient in (26) has the same order of magnitude so the radius of convergence of the series is independent of αR.

4. Conclusions

We strongly emphasize the tentative nature of the results contained in § 3 with regard to the way in which we have defined e_{min}. We have calculated only the first two terms of a regular perturbation series and ideally many more terms should be calculated before the series is truncated and the smallest positive zero $e = e_{\text{min}}$ is found. Further caution is needed because the perturbations are made from modes which exist for the *circular* pipe flow problem. Thus no information can be gleaned about the possible existence of modes which do not exist when the limit $e \to 0$ is taken, which might give instability for *all* e except $e = 0$.

Bearing the above in mind, there are two principal suggestions which are evident from table 1: first, that given e and R the mode which is most likely to be unstable is the $n = 1$ least-damped wall mode (or the associated distributed mode if R is less than about 15 000); second, that flow in an elliptic pipe will be unstable and a critical Reynolds number will exist if e^2 is larger than about 0·07, i.e. if the length of the minor axis is less than about $96\frac{1}{2}\%$ of the length of the major axis. Thus the cross-section of a pipe may not need to differ very much from a circular shape for the flow to be unstable to infinitesimal disturbances.

I thank Dr P. G. Drazin, who originally suggested this problem to me, and the referees for their helpful comments.

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